THERMAL STABILITY OF CONTINUOUS MEDIA WITH VARIABLE THERMOPHYSICAL CHARACTERISTICS

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A continuous medium with distributed heat sources and variable thermophysical characteristics may be located in a multitude of different thermal states which are steady state points of equilibrium or oscillatory processes. For some configurations of the region occupied by the medium an analysis has been performed of thermal state stability as a function of source intensity, thermophysical properties, boundary conditions, and dimensions of the region. The method of stability analysis is based on reduction of the dimensionality of the problem of infinite dimensions by means of projections of solutions into the space of eigenfunctions. We will present results of calculating the thermal states of a cylinder of finite length and their stability.

1. Formulation of the Problem and Method of Solution. A continuous medium with distributed heat sources occupies one of the regions in space bounded by a parallelepiped, located between two concentric spheres, located between two coaxial circular cylinders, closed by plane faces perpendicular to the cylinder axes, and interacts with the surrounding medium. The radii of the inner spheres and cylinder may equal zero. It is assumed that in regions formed by bodies of rotation the temperature depends solely on the linear coordinates. The problem to be studied is that of thermal stability of the medium as a function of source intensity, thermophysical properties of the medium boundary conditions, and form and size of the region occupied by the medium. The thermal state of the medium is described by the thermal conductivity equation with appropriate boundary and initial conditions [1]:

$$\frac{\partial\Theta}{\partial\tau}=\prod_{n=1}^{3}h_n^{-1}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i^{-2}\prod_{k=1}^{3}h_k\bar{\varkappa}(\Theta)\frac{\partial\Theta}{\partial x_i}\right)+\varphi(\Theta)+\varphi_1(x); \tag{1.1}$$

$$H_{ij}(\Theta,\ \partial\Theta/\partial x_i)=0,\ i=1\text{—}3,\ j=1,\ 2; \tag{1.2}$$

$$\Theta(x,\ 0)=0. \tag{1.3}$$

Here $x = (x_1, x_2, x_3)$; θ, x_i, τ are dimensionless temperature, coordinate i, and time; h_i are metric coefficients; $\Theta(\theta)$ is the source function; $\varphi_1(x)$ is a function of coordinate; $\bar{\varkappa}(\Theta)=\varkappa(\Theta)/\varkappa(0)$ is the ratio of the thermal diffusivity coefficients; $H_{ij}(\Theta, \partial\Theta/\partial x_i)$ is the boundary condition of the third kind of point j and coordinate i.

It is assumed that $\partial\varphi(\Theta)/\partial\Theta>0$, while the functions $\varphi(\Theta)$, $\bar{\varkappa}(\Theta)$ can be represented in the form of series

$$\begin{vmatrix}\varphi(\Theta)\\ \bar{\varkappa}(\Theta)\end{vmatrix}=\sum_{i\geqslant 0}\begin{vmatrix}a_i\\ b_i\end{vmatrix}\Theta^i \tag{1.4}$$

(b_i are constants, a_i are functions of the parameter μ, defined in the vicinity of zero).

Then with consideration of Eq. (1.4), Eq. (1.1) takes on the form

$$\partial\Theta/\partial\tau=G(\Theta,\ \mu,\ \Delta)=L_\mu\Theta+G_1(\Theta,\ \mu,\ \Delta), \tag{1.5}$$

where

$$L_\mu=b_0\prod_{n=1}^{3}h_n^{-1}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i^{-2}\prod_{k=1}^{3}h_k\frac{\partial}{\partial x_i}\right)+a_1(\mu);\quad G(\Theta,\mu,\Delta)=\prod_{n=1}^{3}h_n^{-1}\times$$

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$$\times\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(h_i^{-2}\prod_{k=1}^{3}h_k\sum_{m\geqslant 1}\frac{b_m}{m+1}\frac{\partial}{\partial x_i}\Theta^{m+1}\right)+\sum_{i\geqslant 2}a_i\Theta^i+\Delta(a_0+\varphi_1(x));$$

Δ is found from the relationship $G_1(0,\ 0,\ \Delta) = a_0+\varphi_1(x)$.

In accordance with the theory of the central manifold [2] the infinite-dimensional problem of Eqs. (1.2) and (1.5) at $\Delta = 0$ can be reduced to a space of finite dimensions without loss of information relative to stability of the solutions. The dimensionality of the problem can be reduced by constructing one manifold from the set of central manifolds, or by the projection method of [3, 4], the latter method being extensible to $\Delta \neq 0$.

The sequence of operations in the projection method is to initially construct the functional space of the operator L_μ and to determine the stability of the zeroth solution. Then the solutions of bifurcation problem (1.5) ($\Delta = 0$) and the problem of Eq. (1.5) with the defect ($\Delta \neq 0$) that destroys the bifurcation are projected into this space and the stability of their solutions determined.

The possibility of expanding the projection method to the case $G_1(0,\ \mu,\ \Delta) \neq 0$ follows from the results of [2].

2. Space of the Operator L_μ and Stability of the Zeroth Solution. Analysis of stability of the zeroth solution of Eq. (1.5) with conditions (1.2), (1.3) reduces to a problem of the Sturm–Liouville type, involving determination of the eigenfunctions of the operator L_μ. The spectrum of the operator L_μ consists only of discrete eigenvalues $\sigma_n = a_1 - b_0\lambda_n^2$, n = 1, 2, ..., where λ_n^2 satisfy the equation $(\lambda^2 E - L_0)\Theta(x) = 0$ (E is a unit matrix and $\Theta(x)$ the solution of the equation $L_0\Theta = 0$ which satisfies conditions (1.2)).

We then identify the largest eigenvalue σ_1 with the parameter μ and the zeroth solution is stable if $\mu = a_1 - b_0\lambda_1^2 < 0$. Depending on boundary conditions (1.2) the eigenvalues σ_n may be simple, or multiple, so that each of them corresponds to eigenvectors y_{1n}, y_{2n}, ..., y_{Nn}. The maximum geometric multiplicity of σ_n is N = 8. If N = 1 and σ_n are simple, the stability of the solutions of Eq. (1.5) with conditions (1.2), (1.3) can be carried out in the same manner as in [5].

Gramma–Charlier transforms make it possible to orthogonalize the system of vectors v_{ij} ($1 \leq i \leq N$, $1 \leq j < \infty$) such that to the maximum eigenvalue σ_1 there correspond vectors

$$\bar{y}_{11}=y_{11},\ \bar{y}_{k1}=y_{k1}-\sum_{i=1}^{k-1}\langle y_{k1},\bar{y}_{i1}\rho(x)\rangle\|\bar{y}_{i1}\|^{-1}\bar{y}_{i1}. \tag{2.1}$$

Here $\langle\bar{y}_{ij},\bar{y}_{km}\rangle$ is the scalar product of the vectors $\bar{y}_{ij}$, $\bar{y}_{km}$; $\rho(x)$ is a weight function, dependent on the coordinate system chosen; $\|\bar{y}_{ij}\| = \langle\bar{y}_{ij},\bar{y}_{ij}\rho(x)\rangle$. The space of the vectors $\bar{y}_{kn}$ ($1 \leq k \leq 8$, $n \geq 1$) is a Gilbert space with scalar product $\langle(\bar{y}_{1i},\bar{y}_{2j}),(\bar{y}_{1n},\bar{y}_{2m})\rangle=\langle\bar{y}_{1i},\bar{y}_{1n}^*\rangle+\langle\bar{y}_{2j},\bar{y}_{2m}^*\rangle$ ($\bar{y}_{kn}^*$ are eigenvectors of the conjugate operator L_μ^*).

Before analyzing the bifurcation solution of Eq. (1.5), we should make clear that the analysis results are valid only for the case where the algebraic multiplicity factor of the eigenvalues σ_n does not exceed the geometric, although the projection method is also applicable to that case.

3. Bifurcation Solution. The stability of the bifurcation solution can be carried out in a functional space for which N = 2.

The solution of Eq. (1.5) with conditions (1.2), (1.3) for $\Delta = 0$ can be found in the form of series

$$\begin{vmatrix}\Theta\\ \mu\end{vmatrix}=\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\begin{vmatrix}\Theta_n\\ \mu_n\end{vmatrix}, \tag{3.1}$$

where $\varepsilon = \langle(\Theta,\Theta),(\bar{y}_{11},\bar{y}_{21})\rangle$ is the amplitude; Θ_n, μ_n are coefficients of the expansion requiring determination.

Substituting Eq. (3.1) in Eq. (1.5) and equating terms with independent powers of ε leads to the equations

$$L_0\Theta_1 = 0; \tag{3.2}$$

$$L_0\Theta_2 + 2\mu_1 \frac{\partial L_0}{\partial \mu}\Theta_1 + \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\Theta_1^2 = 0, \tag{3.3}$$

plus equations for higher powers of ε.

It follows directly from Eq. (3.2) that the solution can be any linear combination of the vectors $\bar{y}_{11}$, $\bar{y}_{21}$, $\Theta_1 = \bar{y}_{11} + \psi \bar{y}_{21}$, where ψ is a problem parameter requiring determination.

Equation (3.3) is soluble when and only when for k = 1, 2 the conditions $\langle L_0\Theta_2, \bar{y}^*_{k1}\rangle = 0$ are satisfied, these following from the Fredholm alternative theorem. Hence it follows that

$$2\mu_1\left\langle \frac{\partial L_0}{\partial \mu}\Theta_1, \bar{y}^*_{k1}\right\rangle + \left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\Theta_1^2, \bar{y}^*_{k1}\right\rangle = 0, \quad k = 1, 2. \tag{3.4}$$

The presence of two independent variables μ_1, ψ guarantees the existence of a solution to system (3.4). Substitution in these equations of expressions for Θ_1, $\bar{y}^*_{k1}$ (k = 1, 2) yields two equations of conic sections in the plane (μ_1, ψ):

$$g_1(\mu_1, \psi) = c_{11}\psi^2 + c_{12}\psi + c_{13}\mu_1\psi + c_{14}\mu_1 + c_{15} = 0; \tag{3.5}$$

$$g_2(\mu_1, \psi) = c_{21}\psi^2 + c_{22}\psi + c_{23}\mu_1\psi + c_{24}\mu_1 + c_{25} = 0. \tag{3.6}$$

Here

$$c_{11} = 0{,}5\left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{21}^2, \bar{y}^*_{11}\right\rangle; \quad c_{12} = \left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{11}\bar{y}_{21}, \bar{y}^*_{11}\right\rangle;$$

$$c_{13} = \left\langle \frac{\partial L_0}{\partial \mu}\bar{y}_{21}, \bar{y}^*_{11}\right\rangle; \quad c_{14} = \left\langle \frac{\partial L_0}{\partial \mu}\bar{y}_{11}, \bar{y}^*_{11}\right\rangle;$$

$$c_{15} = 0{,}5\left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{11}^2, \bar{y}^*_{11}\right\rangle; \quad c_{21} = 0{,}5\left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{21}^2, \bar{y}^*_{21}\right\rangle;$$

$$c_{22} = \left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{11}\bar{y}_{21}, \bar{y}^*_{21}\right\rangle; \quad c_{23} = \left\langle \frac{\partial L_0}{\partial \mu}\bar{y}_{21}, \bar{y}^*_{21}\right\rangle;$$

$$c_{24} = \left\langle \frac{\partial L_0}{\partial \mu}\bar{y}_{11}, \bar{y}^*_{21}\right\rangle; \quad c_{25} = 0{,}5\left\langle \frac{\partial^2 G(0,0,0)}{\partial \Theta^2}\bar{y}_{11}^2, \bar{y}^*_{21}\right\rangle.$$

In view of the transforms of Eq. (2.1) $c_{13} = c_{24} = 0$, and if Eqs. (3.5), (3.6) are not degenerate then Eq. (3.5) is always a parabola and Eq. (3.6) is always a hyperbola. The points of intersection of curves (3.5), (3.6) $(\mu_1^{(n)}, \psi^{(n)})$ (n = 1-3) in the plane (μ_1, ψ) are solutions of Eq. (3.3). Depending on the sign of the discriminant of the cubic equation equivalent to system (3.5), (3.6)

$$B_3\psi^3 + B_2\psi^2 + B_1\psi + B_0 = 0\,, \tag{3.7}$$

$(B_3 = 1,\quad B_2 = (c_{12} - c_{14}c_{21}c_{23}^{-1})c_{11}^{-1},\quad B_1 = (c_{15} - c_{14}c_{22}c_{23}^{-1})\ c_{11}^{-1},\ B_0 = -c_{14}c_{25}c_{23}^{-1}c_{11}^{-1})$; the system of Eqs. (3.5), (3.6) has either three real solutions, or one real, and two complex-conjugate. If the discriminant is equal to zero, then two or all three of the real roots coincide.

The stability of the solution of Eq. (1.5) must be analyzed at each intersection point of curves (3.5), (3.6) $(\mu_1^{(n)}, \psi^{(n)})$, n = 1-3. To do this it is necessary to write the functions $g_i(\mu_1, \psi)$ (i = 1, 2) in the form of functions of the parameter μ. Combining Eqs. (3.1), (3.5), (3.6) and using the normalization condition ε = 1, we write Eqs. (3.5), (3.6) in the form

$$\bar{g}_i(\mu) = \mu^2\left(c_{i1}\psi^2\mu_1^{-2} + c_{i2}\psi\mu_1^{-2} + c_{i3}\psi\mu_1^{-1} + c_{i4}\mu_1^{-1} + c_{i5}\mu_1^{-2}\right) = 0,\ i = 1, 2. \tag{3.8}$$

Lyapunov's theorem on stability in the first approximation [6] states that Eq. (1.5) is stable, if the real portions of the eigenvalues of the Jacobi matrix

$$I=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\ \left(a_{11}=\partial\bar{g}_1(\mu)/\partial\mu_1^{-1},\ a_{12}=\partial\bar{g}_1(\mu)/\partial\left(\psi\mu_1^{-1}\right),\ a_{21}=\partial\bar{g}_2(\mu)/\partial\mu_1^{-1},\ a_{22}=\partial\bar{g}_2(\mu)/\partial\left(\psi\mu_1^{-1}\right)\right)$$

are negative. A strict proof of this assertion with consideration of attraction of the solution from R^∞ to R^2 was presented in [3, 7].

Considering that at each point $(\mu_1^{(n)}, \psi^{(n)})$ for small μ we have $\det I = \mu^2 \det I(\mu_1^{(n)}, \psi^{(n)}) + O|\mu^3|$, we write the stability condition for steady state equilibrium

$$\max\left(\mu s_1^{(n)}, \mu s_2^{(n)}\right)<0,\ \det I\left(\mu_1^{(n)}, \psi^{(n)}\right)>0; \tag{3.9}$$

and for periodic cycles

$$\begin{gathered}\max\left(\mu \operatorname{Re} s_1^{(n)},\ \mu \operatorname{Re} s_2^{(n)}\right)<0,\\ \left|\operatorname{Re}\left(a_{11}^{(n)}+a_{22}^{(n)}\right)\right|>\left|\left(\alpha_n^2+\beta_n^2\right)^{0,25}\cos\operatorname{arctg}\alpha_n^{-1}\beta_n\right|.\end{gathered} \tag{3.10}$$

Here $s_1^{(n)}$, $s_2^{(n)}$ are eigenvalues of the matrix $I(\mu_1^{(n)}, \psi^{(n)})$;

$$\begin{gathered}a_{ij}^{(n)}=a_{ij}\left(\mu_1^{(n)}, \psi^{(n)}\right);\ \ \alpha_n=\left(\operatorname{Re}\left(a_{11}^{(n)}-a_{22}^{(n)}\right)\right)^2-\left(\operatorname{Im}\left(a_{11}^{(n)}-a_{22}^{(n)}\right)\right)^2+\\ +4\operatorname{Re}a_{12}^{(n)}\operatorname{Re}a_{21}^{(n)}-4\operatorname{Im}a_{12}^{(n)}\operatorname{Im}a_{21}^{(n)};\\ \beta_n=2\operatorname{Re}\left(a_{11}^{(n)}-a_{22}^{(n)}\right)\operatorname{Im}\left(a_{11}^{(n)}-a_{22}^{(n)}\right)+4\operatorname{Re}a_{12}^{(n)}\operatorname{Im}a_{21}^{(n)}+4\operatorname{Re}a_{21}^{(n)}\operatorname{Im}a_{12}^{(n)}.\end{gathered}$$

Condition (3.9) is valid only for transverse intersection of curves (3.5), (3.6) in the plane (μ_1, ψ). This requires that $\det I_0 \neq 0$, where

$$I_0=\begin{vmatrix}\partial g_1(\mu_1,\psi)/\partial\mu_1 & \partial g_1(\mu_1,\psi)/\partial\psi\\ \partial g_2(\mu_1,\psi)/\partial\mu_1 & \partial g_2(\mu_1,\psi)/\partial\psi\end{vmatrix}.$$

If $\det I_0 = 0$, then at the point of intersection of curves (3.5), (3.6) a common tangent exists and higher order approaches are required to study stability. In the presence of only one real root of Eq. (3.7) the intersection is always abrupt, since if the curves are tangent we already have two real roots.

Common to all real solutions is instability if $\det I < 0$, and stability to one side of the point $\mu = 0$ if $\det I > 0$. Since two consecutive points of intersection on the arcs of the conic solutions (3.5), (3.6) have opposite signs of det I, one solution is always unstable for any μ, while the other is stable to one side of the point $\mu = 0$. Hence it follows that if the system can be in several steady thermal states, among those there will always be stable and unstable ones.

For complex solutions of Eqs. (3.5), (3.6) one must consider the special case of Eq. (3.10) $\max(\mu \operatorname{Re} s_1^{(n)}, \mu \operatorname{Re} s_2^{(n)}) = 0$ (the null being simple), at which bifurcation of generation of the limiting cycle occurs – Hopf bifurcation [2].

As an example we will consider a cylinder occupying the region $0 \leq x_1 \leq \ell$, $0 \leq x_2 \leq 2\pi$, $0 \leq x_3 \leq r$. The matrix coefficients $h_1 = 1$, $h_2 = x_3$, $h_3 = 1$. The boundary conditions of Eq. (1.3) can be written as

$$\begin{gathered}\frac{\partial\Theta}{\partial x_1}+\alpha_{11}\Theta\,|_{x_1=0}=0,\ \ \frac{\partial\Theta}{\partial x_1}+\alpha_{12}\Theta\,|_{x_1=l}=0,\\ \Theta\,|_{x_3=0}<\infty,\ \ \frac{\partial\Theta}{\partial x_3}+\alpha_{32}\Theta\,|_{x_3=r}=0.\end{gathered}$$

With such boundary conditions the eigenvalues of the operator L_μ are doubled. The maximum eigenvalue and the corresponding vectors are equal to $\sigma_1 = a_1 - b_0\lambda_1^2$, $\lambda_1^2 = \eta_1^2 +$

$\nu_1^2 r^{-2}$, $y_{11} = I_0(\nu_1 r^{-1} x_3) \sin \eta_1 x_1$, $y_{21} = I_0(\nu_1 r^{-1} x_3) \cos \eta_1 x_1$, where $I_i(\tau_1 r^{-1} x_3)$ are Bessel functions of the first sort and order i; η_1, ν_1 are the smallest positive roots of the equations

$$\operatorname{ctg} \eta_1 l = \frac{1}{\alpha_{12} - \alpha_{11}} \left(\eta_1 + \frac{\alpha_{11}\alpha_{12}}{\eta_1} \right), \quad r\alpha_{23} I_0(\nu_1) - \nu_1 I_1(\nu_1) = 0.$$

The orthogonalized vectors of the operators L_μ, L_μ^* corresponding to σ_1 have the form

$$\bar{y}_{11} = y_{11}, \ \bar{y}_{21} = y_{21} - \langle y_{21}, \ \bar{y}_{11} \ x_3 \rangle \|\bar{y}_{11}\|^{-1} \bar{y}_{11},$$
$$\bar{y}_{11}^* = \bar{y}_{11} x_3 \left(\sum_{i=1}^{2} \|\bar{y}_{i1}\| \right)^{-1}, \quad \bar{y}_{21}^* = \bar{y}_{21} x_3 \left(\sum_{i=1}^{2} \| y_{i1} \| \right)^{-1}$$

($x_3 = \rho(x)$ is the weight function mentioned above).

Data for calculation were chosen with the goal of completely encompassing possible thermal states of the cylinder. For the source function the expression of [8] $\varphi(\Theta) = \exp(\Theta(1 + \beta\Theta)^{-1})$ was used, with the first three terms in its expansion being $a_0 = 1$, $a_1 = 1$, $a_2 = 0.5 - \beta$ (where β is a parameter). Calculations performed for various values of ℓ, b_1 for fixed $\alpha_{11} = 1$, $\alpha_{12} = \alpha_{32} = 0$, $\beta = 0$, $b_0 = 1$ yielded results valid for any r.

1. $b_1 = 0.1$, $\ell = 1.1$. The solutions of Eqs. (3.5), (3.6) are points $(\mu_1^{(1)}; \psi^{(1)}) = (-0.783; -14.556)$, $(\mu_1^{(2)}; \psi^{(2)}) = (8.18 \cdot 10^{-2} - 0.567i; 0.106 + 1.860i)$. The third point is the complex conjugate of the second, so will not be considered further. These points correspond to eigenvalues of the Jacobs matrix $(s_1^{(1)}; s_2^{(1)}) = (7.80 \cdot 10^{-2}; -2.521)$, $(s_1^{(2)}; s_2^{(2)}) = (9.93 \cdot 10^{-2} - 0.193i; -0.283 - 0.154i)$, which indicates that all three solutions are unstable to either side of the point $\mu = 0$.

2. $b_1 = 0.1$, $\ell = 1.267$. We have one real and two complex solutions $(\mu_1^{(1)}; \psi^{(1)}) = (-1.103; -11.154)$, $(\mu_1^{(2)}; \psi^{(2)}) = (5.27 \cdot 10^{-2} - 0.336i; -4.27 \cdot 10^{-4} + 1.729i)$, corresponding to $(s_1^{(1)}; s_2^{(1)}) = (0.178; -1.771)$, $(s_1^{(2)}; s_2^{(2)}) = (-0.122i; -0.247 - 0.167i)$. The real solution is unstable for any μ, while the periodic solution is a limiting cycle (Fig. 1, curve 1).

3. $b_1 = 0.1$, $\ell = 1.5$. The solutions $(\mu_1^{(1)}; \psi^{(1)}) = (-1.382; -8.689)$, $(\mu_1^{(2)}; \psi^{(2)}) = (-1.20 \cdot 10^{-2} - 0.103i; -0.216 + 1.587i)$ correspond to $(s_1^{(1)}; s_2^{(1)}) = (0.397; -1.186)$, $(s_1^{(2)}; s_2^{(2)}) = (-1.02 \cdot 10^{-2} - 0.229i; -0.338 - 7.84 \cdot 10^{-3}i)$. The stationary solution is unstable (Fig. 2, curve 1), while the periodic one is stable for $\mu > 0$ (Fig. 1, curve 2, Fig. 3, curve 1).

4. $b_1 = 0.1$, $\ell = 1.54$. A stationary equilibrium point exists $(\mu_1^{(1)}; \psi^{(1)}) = (-1.436; -8.478)$ together with oscillatory regimes $(\mu_1^{(2)}; \psi^{(2)}) = (-2.74 \cdot 10^{-2} - 7.01 \cdot 10^{-2} i; -0.262 + 1.560i)$. For these $(s_1^{(1)}; s_2^{(1)}) = (0.451; -1.127)$, $(s_1^{(2)}; s_2^{(2)}) = (-0.234i; -0.366 + 2.25 \cdot 10^{-3}i)$, so that the stationary solution is unstable (Fig. 2, curve 2), and the periodic one is a stable limiting cycle (Fig. 3, curve 2).

5. $b_1 = 0.1$, $\ell = 1.6$. For the solutions $(\mu_1^{(1)}; \psi^{(1)}) = (-1.532; -8.280)$, $(\mu_1^{(2)}; \psi^{(2)}) = (5.33 \cdot 10^{-2} - 2.55 \cdot 10^{-2}i; -0.335 + 1.515i)$ we have $(s_1^{(1)}; s_2^{(1)}) = (0.547; -1.058)$, $(s_1^{(2)}; s_2^{(2)}) = (1.30 \cdot 10^{-2} - 0.244i; -0.406 + 1.73 \cdot 10^{-2}i)$. Both regimes (the steady state of Fig. 2, curve 3 and the periodic of Fig. 3, curve 3) are unstable.

6. $b_1 = 0.1$, $\ell = 2$. For the stationary point $(\mu_1^{(1)}; \psi^{(1)}) = (-4.043; -13.457)$ and the periodic regime $(\mu_1^{(2)}; \psi^{(2)}) = (-0.268 + 0.107i; -0.728 + 0.952i)$ we have $(s_1^{(1)}; s_2^{(1)}) = (3.119; -1.876)$, $(s_1^{(2)}; s_2^{(2)}) = (0.101 - 0.346i; -0.628 + 6.48 \cdot 10^{-2}i)$. All regimes are unstable. The periodic solution is shown in Fig. 1 (curve 3).

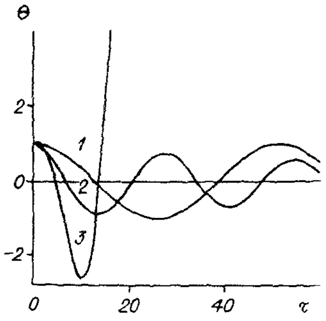

Fig. 1

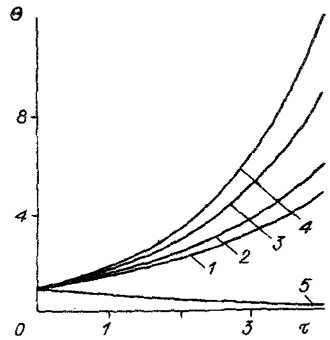

Fig. 2

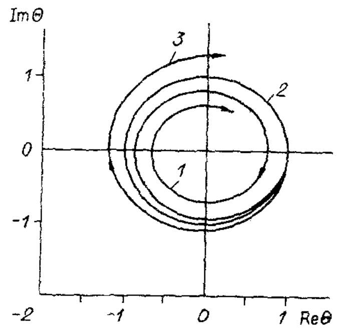

Fig. 3

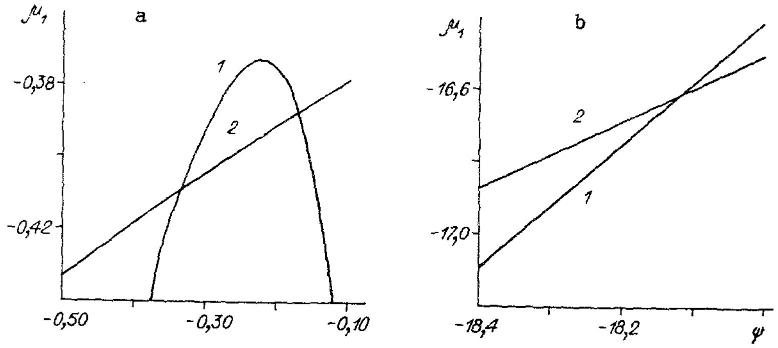

Fig. 4

7. $b_1 = 0.1$, $\ell = 3.5$. All three solutions are steady state equilibrium points $(\mu_1^{(1)}; \psi^{(1)}) = (-16.613; -18.119)$, $(\mu_1^{(2)}; \psi^{(2)}) = (-0.387; -0.169)$, $(\mu_1^{(3)}; \psi^{(3)}) = (-0.409; -0.335)$ (Fig. 4a, b). For these $(s_1^{(1)}; s_2^{(1)}) = (84.337; -19.206)$, $(s_1^{(2)}; s_2^{(2)}) = (0.571; -0.618)$, $(s_1^{(3)}; s_2^{(3)}) = (1.030; 0.359)$. The first two points are unstable for any μ, while the third is stable for $\mu < 0$. The solutions for $(\mu_1(^2); \psi(^2))$, $(\mu_1(^3); \psi(^3))$ are shown in Fig. 2 (curves 4, 5).

8. $b_1 = 0$, $\ell = 1.5$. Of three stationary equilibrium points $(\mu_1^{(1)}; \psi^{(1)}) = (1.348; 4.897)$, $(\mu_1^{(2)}; \psi^{(2)}) = (-1.145; -4.475)$, $(\mu_1^{(3)}; \psi^{(3)}) = (-0.490; -1.139)$ (Fig. 5) one is stable $(s_1{}^{(2)}; s_2{}^{(2)}) = (0.388; 0.189)$ for $\mu < 0$ and two are unstable $(s_1^{(1)}; s_2^{(1)}) = (0.381; -0.411)$, $(s_1^{(3)}; s_2^{(3)}) = (0.326; -8.61 \cdot 10^{-2})$ for any μ. In Figs. 4, 5 curves 1 correspond to Eq. (3.5) and curves 2 to Eq. (3.6).

4. <u>Isolated Solutions</u>. In the general case, for example, $\varphi(\Theta)$ obeys the Arrehnius law, the operator $G(\Theta, \mu, \Delta)$ contains a parameter $\Delta \neq 0$ which destroys the bifurcation at the point $(\Theta, \mu) = 0, 0$, so that the solutions which branch at this point decay into isolated solutions. For $\Delta = 0$ the solution $\Theta = 0$ of the equation $G(\Theta, \mu, 0) = 0$ always loses

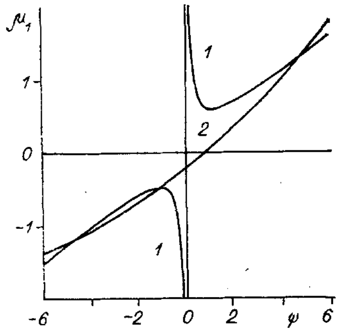

Fig. 5

stability upon transition of μ through zero. From this, it follows by Hopf's assertion [2] that the point $(\Theta, \mu) = (0, 0)$ is a double bifurcation point.

The inequality $\langle \partial G(0, 0, 0)/\partial\Delta, \bar{y}^*_{k1}\rangle \neq 0$ $(k = 1, 2)$ and the implicit function theorem guarantee the existence of a solution $G(\Theta, \mu, \Delta) = 0$ for $\Delta = \Delta(\mu, \varepsilon)$, which can be obtained in the form of a series in powers of μ, ε.

Twofold differentiation of $G(\Theta, \mu, \Delta)$ with respect to μ, ε at the point $(\mu, \varepsilon) = (0, 0)$ and use of the identity $\partial G(\Theta, \mu, \Delta(\mu, 0))/\partial\varepsilon = 0$ which follows from the definition of the double point yields a system of equations

$$\frac{\partial G(0,0,0)}{\partial\Theta}\frac{\partial^2\Theta}{\partial\varepsilon^2} + \frac{\partial^2 G(0,0,0)}{\partial\Theta^2}\Theta_1^2 + \frac{\partial G(0,0,0)}{\partial\Delta}\frac{\partial^2\Delta}{\partial\varepsilon^2} = 0; \tag{4.1}$$

$$\frac{\partial G(0,0,0)}{\partial\Theta}\frac{\partial^2\Theta}{\partial\mu\,\partial\varepsilon} + \frac{\partial^2 G(0,0,0)}{\partial\mu\,\partial\Theta}\Theta_1 + \frac{\partial G(0,0,0)}{\partial\Delta}\frac{\partial^2\Delta}{\partial\mu\,\partial\varepsilon} = 0, \tag{4.2}$$

which is soluble when and only when for $k = 1, 2$ we have the condition

$$\left\langle \frac{\partial^2\Theta}{\partial\varepsilon^2}, \bar{y}^*_{k1}\right\rangle = \left\langle \frac{\partial^2\Theta}{\partial\mu\,\partial\varepsilon}, \bar{y}^*_{k1}\right\rangle = 0,$$

which together with Eqs. (4.1), (4.2) allows finding the first two non-zero terms in the expansion of the function $\Delta(\mu, \varepsilon)$ in powers of μ, ε:

$$\Delta(\mu,\varepsilon) = -\frac{1}{2}\left[\frac{\langle \partial^2 G(0,0,0)/\partial\Theta^2\Theta_1^2, \bar{y}^*_{k1}\rangle}{\langle \partial G(0,0,0)/\partial\Delta, \bar{y}^*_{k1}\rangle}\varepsilon^2 + 2\frac{\langle \partial^2 G(0,0,0)/\partial\Theta\partial\mu\Theta_1, \bar{y}^*_{k1}\rangle}{\langle \partial G(0,0,0)/\partial\Delta, \bar{y}^*_{k1}\rangle}\mu\varepsilon\right], \quad k = 1, 2. \tag{4.3}$$

Equations (4.3) define isolated solutions of Eqs. (1.2), (1.3), (1.5) in the plane (μ_1, ψ). Substitution of expressions for Θ_1, $\bar{y}^*_{k1}$ $(k = 1, 2)$ in Eq. (4.3) and application of the normalization condition $\varepsilon = 1$ leads to the system

$$g_1(\mu_1, \psi) + \Delta\langle a_0 + \varphi_1(x), \bar{y}^*_{11}\rangle = 0; \tag{4.4}$$

$$g_2(\mu_1, \psi) + \Delta\langle a_0 + \varphi_1(x), \bar{y}^*_{21}\rangle = 0, \tag{4.5}$$

where $g_i(\mu_1, \psi)$ $(i = 1, 2)$ are found from Eqs. (3.5), (3.6). The coefficients of the cubic equation equivalent to system (4.4), (4.5) are as follows:

$$B_0 = -\left(c_{25} + \Delta\langle a_0 + \varphi_1(x), \bar{y}^*_{21}\rangle\right)c_{14}c_{11}^{-1}c_{23}^{-1},$$
$$B_1 = \left(c_{15} - c_{14}c_{22}c_{23}^{-1} + \Delta\langle a_0 + \varphi_1(x), \bar{y}^*_{11}\rangle\right)c_{11}^{-1}$$

(B_2, B_3 are the same as in Eq. (3.7)).

The system of Eqs. (4.4), (4.5) does not differ in structure from Eqs. (3.5), (3.6), so that the further analysis of stability of the solutions of Eq. (1.5) with conditions (1.2), (1.3) is analogous to that presented in section 3 for the bifurcation solution. The distribution of real and complex solutions of Eq. (1.5) for $\Delta \neq 0$ is of the same form as for $\Delta = 0$, although not necessarily identical.

Assuming that the properties of the material and the region which it occupies are specified, it makes sense to consider boundary conditions (1.2) and the function $\varphi_1(x)$ as types of regulation instruments which allow control of the distribution of thermal states of the medium and their strength as attractors.

Thus, according to the results obtained, the presence of a set of thermal states, both steady and periodic, in which the medium can be found stimulates the thought that uncontrolled thermal processes (explosions, fires) which occur during processing, storage, and accumulation of materials in the chemical, atomic, coal, petroleum, and milling industries, etc. may occur not only because of breaking of rules (metric, mass, temperature, concentration), but also in a "legal" manner, if such rules are developed without consideration of all possible thermal states.

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